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Krall-type polynomials via the Heine formula

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Abstract
With the Heine multiple integral formula we study the effect of a special class of perturbations, adding δ and δ' to the weights, on the Laguerre and Jacobi polynomials.

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1. Introduction

Non-classical polynomials which are eigenfunctions of a certain fourth-order differential equation were classified in the pioneering work of [7]. These polynomials are orthogonal with respect to a class of specially perturbed classical weights. These perturbations consist of adding a mass point(s) to the Laguerre and Jacobi weights, localized on the boundary point(s) of the interval of orthogonality. It was proved in [2] that such polynomials can be obtained through a Darboux transformation on the original three-term recurrence relations. From this point of view, the recurrence relation becomes a discrete Schrödinger equation, governing, for example, the motion of an electron in a one-dimensional lattice. By iterating the Darboux transformation, α times, it was shown in [3], starting from the Laguerre case, that the polynomials thus obtained are orthogonal with respect to $x^{\alpha-k}\exp(-x)/\Gamma(\alpha-k+1)+\sum_{j=1}^{k}s_j\delta^{(k-j)}(x)$, where $x\in[0,\infty)$ and $1\leqslant k\leqslant\alpha$. It was also shown due to the rational character of the Darboux transformation that these polynomials are also eigenfunctions of a finite-order differential operator, with order greater than two. For further development see the papers cited in [3] and also the recent review [4]. The reader may also find more recent material in this area in a special volume of *J. Comput. Appl. Math.* **133** August 2001.

In this paper, following a suggestion of Haine, we adopt a different approach to this problem. Instead we apply directly the multiple integral representation of Heine [5] to a special class of perturbed weights.

In this section, we obtain explicit expressions for the Krall–Laguerre and Koornwinder polynomials [6]. In section 2, we describe, using the Heine formula, the effect of adding δ' to a general weight. In sections 3 and 4, we specialize to the Laguerre and Jacobi weights and give explicit formulae for the respective polynomials. In section 5 we determine the large n asymptotics of the appropriately scaled Christoffel–Darboux kernel. In section 6, the asymptotic zero distribution of the transformed polynomials is determined from a second-order differential equation satisfied by them.

The Hankel determinant of an $n \times n$ moment matrix, μ_{i+j}, is

$$\mathcal{D}_n[w(x)] := \det_{0\leqslant i,j\leqslant n-1}(\mu_{i+j}) = \frac{1}{n!}\left(\prod_{l=1}^{n}\int_J \mathrm{d}x_l w(x_l)\right)\prod_{1\leqslant j<k\leqslant n}|x_j - x_k|^2 \tag{1.1}$$

where

$$\mu_j := \int_J x^j w(x)\,\mathrm{d}x. \tag{1.2}$$

A theorem of Heine [5], states that the monic polynomials, $p_n(t, [w])$, orthogonal with respect to w supported on J, i.e.

$$\int_J p_m(t, [w])p_n(t, [w])w(t)\,\mathrm{d}t = h_n\delta_{mn} \tag{1.3}$$

are given by

$$p_n(t, [w]) = \frac{1}{\mathcal{D}_n[w]\, n!}\left(\prod_{l=1}^{n}\int_J \mathrm{d}x_l w(x_l)(t - x_l)\right)\prod_{1\leqslant j<k\leqslant n}|x_j - x_k|^2. \tag{1.4}$$

Note that $\mathcal{D}_n[w] = \prod_{j=0}^{n-1} h_j$. The Barnes G function [1] is defined by the functional equation $G(z + 1) = \Gamma(z)G(z)$ with the condition $G(1) = 1$. Therefore $\prod_{j=0}^{n-1}\Gamma(j + z) = G(z + n)/G(z)$. This will be useful later.

In what follows we use (1.1) and (1.4) to examine the effect on a general set of polynomials and their associated Hankel determinants when we add an extra term to the weight $w(x)$. Initially we consider the weight function $w(x) + A\delta(x - a)$, where A is a real constant and $a \in J$. From (1.1),

$$\begin{aligned}\mathcal{D}_n[w(x) + A\delta(x - a)] &= \frac{1}{n!}\left(\prod_{l=1}^{n}\int_J \mathrm{d}x_l[w(x_l) + A\delta(x_l - a)]\right)\prod_{1\leqslant j<k\leqslant n}|x_j - x_k|^2\\ &= \mathcal{D}_n[w(x)] + \frac{A}{(n-1)!}\int_J \mathrm{d}x_n\delta(x_n - a)\left(\prod_{l=1}^{n-1}\int_J \mathrm{d}x_l w(x_l)\right)\prod_{1\leqslant j<k\leqslant n}|x_j - x_k|^2\\ &= \mathcal{D}_n[w(x)] + A\mathcal{D}_{n-1}[w(x)(x - a)^2].\end{aligned} \tag{1.5}$$

It follows that, for $a,\ b \in J$,

$$\begin{aligned}\mathcal{D}_n[w(x) + A\delta(x - a) + B\delta(x - b)] &= \mathcal{D}_n[w(x)] + A\mathcal{D}_{n-1}[w(x)(x - a)^2]\\ &\quad + B\mathcal{D}_{n-1}[w(x)(x - b)^2] + AB(b - a)^2\mathcal{D}_{n-2}[w(x)(x - a)^2(x - b)^2].\end{aligned} \tag{1.6}$$

Similarly,

$$\begin{aligned}p_n(t, [w(x) + A\delta(x - a)]) &= \frac{p_n(t, [w])}{1 + A\mathcal{D}_{n-1}[w(x)(x - a)^2]/\mathcal{D}_n[w]}\\ &\quad + A(t - a)\frac{p_{n-1}(t, [w(x)(x - a)^2])}{A + \mathcal{D}_n[w]/\mathcal{D}_{n-1}[w(x)(x - a)^2]}.\end{aligned} \tag{1.7}$$

Also,

$$\begin{aligned}\mathcal{D}_n[W_{A,B}(x)]p_n(t, [W_{A,B}(x)]) &= \mathcal{D}_n[w(x)]p_n(t, [w(x)])\\ &\quad + A(t - a)\mathcal{D}_{n-1}[w(x)(x - a)^2]p_{n-1}(t, [w(x)(x - a)^2])\end{aligned}$$

$$
\begin{aligned}
&+ B(t-b)\mathcal{D}_{n-1}[w(x)(x-b)^2]p_{n-1}(t,[w(x)(x-b)^2]) \\
&+ AB(t-a)(t-b)(b-a)^2\mathcal{D}_{n-2}[w(x)(x-a)^2(x-b)^2] \\
&\times p_{n-2}(t,[w(x)(x-a)^2(x-b)^2])
\end{aligned} \tag{1.8}
$$

where

$$W_{A,B}(x) := w(x) + A\delta(x-a) + B\delta(x-b).$$

Equations (1.5) and (1.7) are of particular interest for the classical monic Laguerre and Jacobi polynomials defined by their respective orthogonality relations

$$\int_0^\infty t^\alpha e^{-t} L_n^\alpha(t) L_m^\alpha(t)\,dt = \delta_{nm} h_n^\alpha \qquad h_n^\alpha = \Gamma(n+1)\Gamma(n+\alpha+1) \tag{1.9}$$

and

$$\int_{-1}^1 (1-t)^\alpha (t+1)^\beta P_n^{\alpha,\beta}(t) P_m^{\alpha,\beta}(t)\,dt = \delta_{nm} h_n^{\alpha,\beta} \tag{1.10}$$

$$h_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1+2n}\Gamma(n+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)[\Gamma(2n+\alpha+\beta+1)]^2}.$$

In the case of Laguerre, taking $w(x) = x^\alpha e^{-x}$, and putting $a=0$, in (1.7), we find,

$$p_n^\alpha(t,[x^\alpha e^{-x} + A\delta(x)]) = \frac{L_n^\alpha(t)}{1 + A\mathcal{D}_{n-1}^{\alpha+2}/\mathcal{D}_n^\alpha} + At\frac{L_{n-1}^{\alpha+2}(t)}{A + \mathcal{D}_n^\alpha/\mathcal{D}_{n-1}^{\alpha+2}}. \tag{1.11}$$

In terms of the Barnes G-function, the Hankel determinant for the Laguerre case is

$$\mathcal{D}_n^\alpha := \mathcal{D}_n[x^\alpha e^{-x}] = \frac{G(n+1)G(n+\alpha+1)}{G(\alpha+1)}. \tag{1.12}$$

Similarly for Jacobi, we set $a = -1$ and (1.7) becomes

$$p_n^{\alpha,\beta}(t,[(1-x)^\alpha(x+1)^\beta + A\delta(x+1)]) = \frac{P_n^{\alpha,\beta}(t)}{1 + A\mathcal{D}_{n-1}^{\alpha,\beta+2}/\mathcal{D}_n^{\alpha,\beta}} + A(t+1)\frac{P_{n-1}^{\alpha,\beta+2}(t)}{A + \mathcal{D}_n^{\alpha,\beta}/\mathcal{D}_{n-1}^{\alpha,\beta+2}} \tag{1.13}$$

where

$$
\begin{aligned}
\mathcal{D}_n^{\alpha,\beta} &= \mathcal{D}_n[(1-x)^\alpha(x+1)^\beta] \\
&= \frac{2^{n(\alpha+\beta+n)}G(n+1)G(n+\alpha+\beta+1)G(n+\alpha+1)G(n+\beta+1)}{G(2n+\alpha+\beta+1)G(\alpha+1)G(\beta+1)}.
\end{aligned} \tag{1.14}
$$

Using (1.8) we can also add a second delta function to the Jacobi weight at $b = 1$ in which case the result of Koornwinder [6] is recovered.

2. Adding δ'

In this section, we give expressions for the transformed polynomials when we add $\delta'(x-a)$ to the weight function. First we derive a general expression for the transformed Hankel determinant. For this we use the simplifying notation, $[jk] := (x_j - x_k)^2$. So

$$
\begin{aligned}
\mathcal{D}_n[w + A\delta'(x-a)] &= \frac{1}{n!}\prod_{i=1}^N \int dx_i(w(x_i) + A\delta'(x_i - a)) \prod_{1\leqslant j,k\leqslant n}[jk] \\
&= \mathcal{D}_n[w] + \varphi_n A + \psi_n A^2.
\end{aligned}
$$

Note that the coefficients of $A^k, k \geqslant 3$ are zero. The coefficient of A is a sum of n terms. These can be grouped together into one term due to the permutation symmetry of the first $n-1$ variables, thus leaving

$$\varphi_n = \frac{1}{(n-1)!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \delta'(x_n - a) \prod_{i=1}^{n-1} w(x_i) \prod_{1\leqslant j,k\leqslant n} [jk]. \tag{2.1}$$

Integrating with respect to x_n, using $\int_p^q \delta'(x-a) f(x)\,\mathrm{d}x = -f'(a)$, for $p < a < q$, we find

$$\varphi_n = -\frac{\partial}{\partial a}\mathcal{D}_{n-1}[w(x)(x-a)^2].$$

The coefficient of A^2, ψ_n is easily evaluated. Therefore the Hankel determinant is

$$\mathcal{D}_n[w(x) + A\delta'(x-a)] = \mathcal{D}_n[w(x)] - A\frac{\partial}{\partial a}\mathcal{D}_{n-1}[w(x)(x-a)^2] - A^2\mathcal{D}_{n-2}[w(x)(x-a)^4]. \tag{2.2}$$

Here φ_n has an alternative representation which is linked to random matrix theory [9]. In random matrices, the following quantity:

$$K_n(x,x;[w]) := \frac{w(x)}{h_{n-1}}\left(p_n'(x)p_{n-1}(x) - p_{n-1}'(x)p_n(x)\right) \tag{2.3}$$

is the eigenvalue density at the point x. This is also the 1-point correlation function

$$R_{1n}(x_1,[w]) := n\int \mathrm{d}x_2 \int \mathrm{d}x_3 \ldots \int \mathrm{d}x_n\ \mathcal{P}_n(x_1,x_2,\ldots,x_n) \tag{2.4}$$

where

$$\mathcal{P}_n(x_1,\ldots,x_n) = \frac{1}{\mathcal{D}_n[w(x)]n!}\prod_{i=1}^{n} w(x_i) \prod_{1\leqslant j,k\leqslant n}[jk]. \tag{2.5}$$

Consequently

$$\frac{\partial}{\partial a}\mathcal{D}_{n-1}[w(x)(x-a)^2] = -2\mathcal{D}_{n-1}[w(x)(x-a)^2]\int_J \frac{1}{x-a}K_n(x,x;w(x)(x-a)^2)\,\mathrm{d}x. \tag{2.6}$$

Therefore, (2.2) becomes

$$\begin{aligned}\mathcal{D}_n[w + A\delta'(x-a)] = {} & \mathcal{D}_n[w(x)] + 2A\mathcal{D}_{n-1}[w(x)(x-a)^2] \\ & \times \int_J \frac{\mathrm{d}x}{x-a}K_{n-1}(x,x;w(x)(x-a)^2) - A^2\mathcal{D}_{n-2}[w(x)(x-a)^4].\end{aligned} \tag{2.7}$$

Now we study the effect on the polynomials of adding $A\delta'(x-a)$ to the weight. First we give a general expression and then specialize to Laguerre and Jacobi weights.

The Heine formula for $p_n(t,[w(x) + A\delta'(x-a)]$ is

$$\begin{aligned}&\mathcal{D}_n[w + A\delta'(x-a)]p_n(t,[w + A\delta'(x-a)]) \\ &\quad = \prod_{l=1}^{n}\int_J \mathrm{d}x_l[w(x_l) + A\delta'(x_l-a)](t-x_l) \prod_{1\leqslant j<k\leqslant n}[jk] \\ &\quad = \mathcal{D}_n[w]p_n(t,[w]) + \lambda_n(t)\,A + \varrho_n(t)\,A^2.\end{aligned}$$

Again the coefficients of $A^k,\ k \geqslant 3$ are zero. The coefficient of A reads

$$\begin{aligned}
\lambda_n(t) &= \frac{1}{(n-1)!}\left(\prod_{l=1}^{n}\int_J \mathrm{d}x_l\right) w(x_1)\ldots w(x_{n-1})\delta'(x_n-a)\left(\prod_{m=1}^{n}(t-x_m)\right)\prod_{1\leqslant j<k\leqslant n}[jk] \\
&= -\frac{1}{(n-1)!}\left(\prod_{l=1}^{n}\int_J \mathrm{d}x_l\right) w(x_1)\ldots w(x_{n-1})\delta(x_n-a)\frac{\partial}{\partial x_n} \\
&\quad\times\left[\left(\prod_{m=1}^{n}(t-x_m)\right)\prod_{1\leqslant j<k\leqslant n}[jk]\right] \\
&= \mathcal{D}_{n-1}[w(x)(x-a)^2]\, p_{n-1}(t,[w(x)(x-a)^2) \\
&\quad -\frac{1}{(n-1)!}(t-a)\left(\prod_{l=1}^{n-1}\int_J \mathrm{d}x_l w(x_l)(t-x_l)\right) \\
&\quad \times\frac{\partial}{\partial a}\left(\prod_{m=1}^{n}(x_m-a)^2\right)\prod_{1\leqslant j<k\leqslant n-1}[jk] \\
&= \mathcal{D}_{n-1}[w(x)(x-a)^2]p_{n-1}(t,[w(x)(x-a)^2]) \\
&\quad -(t-a)\frac{\partial}{\partial a}\mathcal{D}_{n-1}[w(x)(x-a)^2]\, p_{n-1}(t,[w(x)(x-a)^2])
\end{aligned} \tag{2.8}$$

and the coefficient of A^2, $\varrho_n(t)$ can be easily determined.

The polynomials become

$$\begin{aligned}
p_n(t,[w+A\delta'(x-a)]) &= \frac{\mathcal{D}_n[w(x)]}{\mathcal{D}_n[w(x)+A\delta'(x-a)]}p_n(t,[w(x)]) \\
&\quad+\frac{A}{\mathcal{D}_n[w(x)+A\delta'(x-a)]}\Big[\mathcal{D}_{n-1}[w(x)(x-a)^2]p_{n-1}(t,[w(x)(x-a)^2]) \\
&\quad-(t-a)\frac{\partial}{\partial a}\big(\mathcal{D}_{n-1}[w(x)(x-a)^2]p_{n-1}(t,[w(x)(x-a)^2])\big)\Big] \\
&\quad- A^2\frac{\mathcal{D}_{n-2}[w(x)(x-a)^4]}{\mathcal{D}_n[w(x)+A\delta'(x-a)]}(t-a)^2 p_{n-2}(t,[w(x)(x-a)^4]).
\end{aligned} \tag{2.9}$$

Equation (2.9) is the starting point for specializing to the Laguerre and Jacobi weights. However, further study needs to be done as we would like to express $\frac{\partial}{\partial a}(\cdots)$ in (2.9), when a is taken to be a boundary point of the support of the weight, in terms of known polynomials. We proceed as follows:

3. Laguerre $w(x) = x^\alpha \mathrm{e}^{-x}$

For Laguerre, we put $a=0$ in (2.9). This then leaves us with a multiple integral for $\frac{\partial}{\partial a}(\cdots)|_{a=0}$, which will be evaluated below. For Jacobi, however, the corresponding integral is more difficult to evaluate, so we retrace our steps a little to where we calculated the coefficient of A, (2.8), and continue from there.

Thus

$$\begin{aligned}
p_n^\alpha(t,[w(x)+A\delta'(x)] &= \frac{\mathcal{D}_n^\alpha}{\mathcal{D}_n[w(x)+A\delta'(x)]}L_n^\alpha(t)+\frac{A}{\mathcal{D}_n[w(x)+A\delta'(x)]} \\
&\quad\times\left[\mathcal{D}_{n-1}^{\alpha+2}L_{n-1}^{\alpha+2}(t)-t\frac{\partial}{\partial a}\big(\mathcal{D}_{n-1}^{\alpha+2}p_{n-1}^{\alpha+2}(t,[w(x)(x-a)^2])\big)\bigg|_{a=0}\right]
\end{aligned}$$

$$- \frac{A^2\mathcal{D}_{n-2}^{\alpha+4}}{\mathcal{D}_n[w(x) + A\delta'(x)]} t^2 L_{n-2}^{\alpha+4}(t). \tag{3.1}$$

We find

$$\left[\frac{\partial}{\partial a}\left(\mathcal{D}_{n-1}^{\alpha+2} p_{n-1}^{\alpha+2}(t, [w(x)(x-a)^2])\right)\right]_{a=0} = \frac{2I_{n-1}(t)}{(n-2)!} \tag{3.2}$$

where

$$I_n(t) := \int_0^\infty \cdots \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_n x_1^{\alpha+1} \mathrm{e}^{-x_1} x_2^{\alpha+2} \mathrm{e}^{-x_2} \ldots x_n^{\alpha+2} \mathrm{e}^{-x_n} \prod_{l=1}^{n} (t - x_l) \prod_{1 \leqslant j,k \leqslant n} [jk].$$

In order to evaluate this, consider a similar integral

$$J_n(t) := \int_0^\infty \cdots \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_{l=1}^{n} x_l^{\alpha+2} \mathrm{e}^{-x_l} (t - x_l) \prod_{1 \leqslant j,k \leqslant n} [jk] = n! \mathcal{D}_n^{\alpha+2} L_n^{\alpha+2}(t).$$

The substitution $x_l = s_l - a$

$$J_n(t) = \int_a^\infty \cdots \int_a^\infty \mathrm{d}s_1\, \mathrm{d}s_2 \cdots \mathrm{d}s_n \prod_{l=1}^{n} (s_l - a)^{\alpha+2} \mathrm{e}^{-(s_l - a)} (t + a - s_l) \prod_{1 \leqslant j,k \leqslant n} [jk]$$

leaves J_n independent of a. Therefore

$$0 = \frac{\partial J_n}{\partial a} = -(\alpha+2) n I_n + n J_n + \frac{\mathrm{d}J_n}{\mathrm{d}t},$$

and so

$$I_n(t) = \frac{J_n(t)}{\alpha+2} + \frac{1}{n(\alpha+2)} \frac{\mathrm{d}J_n}{\mathrm{d}t} = \frac{n! \mathcal{D}_n^{\alpha+2} L_n^{\alpha+2}(t)}{\alpha+2} + \frac{(n-1)! \mathcal{D}_n^{\alpha+2}}{\alpha+2} \frac{\mathrm{d}}{\mathrm{d}t} L_n^{\alpha+2}(t). \tag{3.3}$$

Therefore the transformed Laguerre polynomials read

$$R(n,\alpha) p_n^\alpha(t, [w + A\delta'(x)]) = L_n^\alpha(t) + A r_1(n,\alpha) \Bigg[L_{n-1}^{\alpha+2}(t) + \frac{2t}{\alpha+2}(n-1) \times \left(L_{n-2}^{\alpha+3}(t) + L_{n-1}^{\alpha+2}(t) \right) \Bigg] - A^2 r_2(n,\alpha) t^2 L_{n-2}^{\alpha+4}(t) \tag{3.4}$$

where we have made use of the fact that, for the monic Laguerre polynomials

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{n-1}^{\alpha+2}(t) = (n-1) L_{n-2}^{\alpha+3}(t).$$

Here

$$r_1(n,\alpha) := \frac{\mathcal{D}_{n-1}^{\alpha+2}}{\mathcal{D}_n^\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)} \sim \frac{n^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+2)}$$

$$r_2(n,\alpha) := \frac{\mathcal{D}_{n-2}^{\alpha+4}}{\mathcal{D}_n^\alpha} = \frac{\Gamma(n+\alpha+2)\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(n-1)\Gamma(\alpha+1)\ldots\Gamma(\alpha+4)} \sim \frac{n^{2\alpha+4}}{\Gamma(\alpha+1)\ldots\Gamma(\alpha+4)}$$

and

$$R(n,\alpha) := \frac{\mathcal{D}_n[x^\alpha \mathrm{e}^{-x} + A\delta'(x)]}{\mathcal{D}_n^\alpha} = 1 + \frac{2A}{\alpha+2}(n-1) r_1(n,\alpha) - A^2 r_2(n,\alpha)$$

$$\sim 1 + \frac{2A}{\alpha+2} \frac{n^{\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)} - A^2 \frac{n^{2\alpha+4}}{\Gamma(\alpha+1)\ldots\Gamma(\alpha+4)}.$$

Using

$$L_{n-1}^{\alpha}(t) = \frac{1}{n}\left(L_n^{\alpha-1}(t) - L_n^{\alpha}(t)\right)$$
$$L_{n-2}^{\alpha}(t) = \frac{1}{n(n-1)}\left(L_n^{\alpha-2}(t) - 2L_n^{\alpha-1}(t) + L_n^{\alpha}(t)\right)$$

we obtain an alternative form for the transformed Laguerre polynomials which will be convenient for the determination of the large n asymptotics in section 5. Therefore (3.4) becomes

$$\begin{aligned} R(n,\alpha)p_n(t) &= L_n^{\alpha}(t) + \frac{A}{n}r_1(n,\alpha)\left(1 + \frac{2n}{\alpha+2}t\right)L_n^{\alpha+1}(t) \\ &\quad - \left(\frac{A}{n}r_1(n,\alpha)\left(1 + \frac{2(n+1)}{\alpha+2}t\right) + \frac{A^2}{n(n-1)}r_2(n,\alpha)t^2\right)L_n^{\alpha+2}(t) \\ &\quad + \left(\frac{2A}{n(\alpha+2)}r_1(n,\alpha)t + \frac{2A^2}{n(n-1)}r_2(n,\alpha)t^2\right)L_n^{\alpha+3}(t) \\ &\quad - \frac{A^2}{n(n-1)}r_2(n,\alpha)t^2 L_n^{\alpha+4}(t). \end{aligned} \tag{3.5}$$

From (2.7) the perturbed Hankel determinant is

$$\mathcal{D}_n[x^{\alpha}\mathrm{e}^{-x} + A\delta'(x)] = \mathcal{D}_n^{\alpha} + 2A\mathcal{D}_{n-1}^{\alpha+2}\int_0^{\infty}\frac{\mathrm{d}x}{x}K_{n-1}(x,x;x^{\alpha+2}\mathrm{e}^{-x}) - A^2\mathcal{D}_{n-2}^{\alpha+4}.$$

The integral appearing in the above equation can be evaluated as follows. Upon the substitution $x_l = s_l - a$, in

$$\mathcal{D}_n^{\alpha+2} = \left(\prod_{l=1}^{n}\int_0^{\infty}\mathrm{d}x_l\, x_l^{\alpha+2}\mathrm{e}^{-x_l}\right)\prod_{1\leqslant j<k\leqslant n}|x_j - x_k|^2$$

we find

$$\mathcal{D}_n^{\alpha+2} = \left(\prod_{l=1}^{n}\int_a^{\infty}\mathrm{d}s_l(s_l - a)^{\alpha+2}\mathrm{e}^{-s_l+a}\right)\prod_{1\leqslant j<k\leqslant n}|s_j - s_k|^2.$$

Now, since $\mathcal{D}_n^{\alpha+2}$ is independent of a

$$\frac{\partial}{\partial a}\mathcal{D}_n^{\alpha+2} = n\mathcal{D}_n^{\alpha+2} - \mathcal{D}_n^{\alpha+2}(\alpha+2)\int_0^{\infty}\frac{\mathrm{d}x}{x}K_{n-1}(x,x;x^{\alpha+2}\mathrm{e}^{-x}) = 0. \tag{3.6}$$

Therefore,

$$\int_0^{\infty}\frac{\mathrm{d}x}{x}K_n(x,x;x^{\alpha+2}\mathrm{e}^{-x}) = \frac{n}{\alpha+2} \tag{3.7}$$

and

$$\mathcal{D}_n[x^{\alpha}\mathrm{e}^{-x} + A\delta'(x)] = \mathcal{D}_n^{\alpha} + \frac{2A}{\alpha+2}(n-1)\mathcal{D}_{n-1}^{\alpha+2} - A^2\mathcal{D}_{n-2}^{\alpha+4}. \tag{3.8}$$

Equation (3.7) can also be obtained by direct integration.

To obtain the recurrence coefficients, we note the recurrence relation

$$zp_n(z) = p_{n+1}(z) + b_{n+1}p_n(z) + a_n p_{n-1}(z).$$

If $c_1(n)$ and $c_2(n)$ are respectively the coefficients of z^{n-1} and z^{n-2} of monic $p_n(z)$, then the recurrence coefficients are given by

$$b_{n+1} = c_1(n) - c_1(n+1)$$
$$a_n = c_2(n) - c_2(n+1) - (c_1(n) - c_1(n+1))c_1(n).$$

From (3.4),

$$R(n,\alpha)c_1^\alpha(n) = l_1^\alpha(n) + Ar_1(n,\alpha)\left(1 + \frac{2}{\alpha+2}(n-1)\left(1 + l_1^{\alpha+2}(n-1)\right)\right) - A^2 r_2(n,\alpha) l_1^{(\alpha+4)}(n-2)$$

$$R(n,\alpha)c_2^\alpha(n) = l_2^\alpha(n) + Ar_1(n,\alpha)\left(l_1^{\alpha+2}(n-1) + \frac{2(n-1)}{\alpha+2}\left(l_1^{\alpha+3}(n-2) + l_2^{\alpha+2}(n-1)\right)\right) - A^2 r_2(n,\alpha) l_2^{(\alpha+4)}(n-2)$$

where $l_{1,2}^\alpha(n)$ are respectively the coefficients of z^{n-1} and z^{n-2} in $L_n^\alpha(z)$;

$$l_1^\alpha(n) = -n(n+\alpha)$$

$$l_2^\alpha(n) = \frac{n(n-1)(n+\alpha-1)(n+\alpha)}{2}.$$

4. Jacobi $w(x) = (1-x)^\alpha(1+x)^\beta$, $x \in [-1, 1]$

As mentioned earlier, the corresponding method for the Jacobi polynomials is more complicated so a slightly different approach is adopted. Instead we consider the set of polynomials orthogonal on $[a, b]$ defined by

$$\int_a^b (b-t)^\alpha (t-a)^\beta p_n^{\alpha,\beta}(t) p_m^{\alpha,\beta}(t)\,\mathrm{d}t = h_n^{\alpha,\beta}(a,b)\delta_{nm}. \tag{4.1}$$

The monic polynomials, $p_n^{\alpha,\beta}$, are related to the monic Jacobi polynomials, $P_n^{\alpha,\beta}$, orthogonal with respect to $(1-t)^\alpha(t+1)^\beta$ on the interval $[-1, 1]$ through

$$p_n^{\alpha,\beta}(t) = \left(\frac{b-a}{2}\right)^n P_n^{\alpha,\beta}(s) \qquad \text{where} \quad s = 2\frac{(t-a)}{(b-a)} - 1. \tag{4.2}$$

Thus we are able to insert the polynomials, $p_n^{\alpha,\beta}$ into equation (2.9) and compute the derivative with respect to a. The only point of caution is that (2.9) was derived assuming the weight function was independent of a. Now that the weight function is a dependent the last step of (2.8) results in a factor of $\frac{2}{\beta+2}$ appearing in the third term. Therefore we have for the polynomials

$$\begin{aligned} p_n^{\alpha,\beta}(t, [w + A\delta'(x-a)]) &= \frac{\mathcal{D}_n^{\alpha,\beta}(a,b)}{\mathcal{D}_n[w + A\delta'(x-a)]}\left(\frac{b-a}{2}\right)^n P_n^{\alpha,\beta}\left(2\frac{(t-a)}{(b-a)} - 1\right) \\ &+ \frac{A}{\mathcal{D}_n[w + A\delta'(x-a)]}\left[\mathcal{D}_{n-1}^{\alpha,\beta+2}(a,b)\left(\frac{b-a}{2}\right)^{n-1} P_{n-1}^{\alpha,\beta+2}\left(2\frac{(t-a)}{(b-a)} - 1\right)\right. \\ &\left. - (t-a)\frac{2}{\beta+2}\frac{\partial}{\partial a}\left(\mathcal{D}_{n-1}^{\alpha,\beta+2}(a,b)\left(\frac{b-a}{2}\right)^{n-1} P_{n-1}^{\alpha,\beta+2}\left(2\frac{(t-a)}{(b-a)} - 1\right)\right)\right] \\ &- A^2\frac{\mathcal{D}_{n-2}^{\alpha,\beta+4}(a,b)}{\mathcal{D}_n[w + A\delta'(x-a)]}(t-a)^2\left(\frac{b-a}{2}\right)^{n-2} P_{n-2}^{\alpha,\beta+4}\left(2\frac{(t-a)}{(b-a)} - 1\right). \end{aligned} \tag{4.3}$$

Denoting $\mathcal{D}_n^{\alpha,\beta}(a,b) = \mathcal{D}_n[(b-x)^\alpha(x-a)^\beta]$, the derivative with respect to a in (4.3) becomes

$$-\left(\frac{b-a}{2}\right)^{n-1}\left(\frac{(t-a)}{(b-a)} - 1\right)\mathcal{D}_{n-1}^{\alpha,\beta+2}(a,b)\frac{\partial}{\partial t}P_{n-1}^{\alpha,\beta+2}\left(2\frac{(t-a)}{(b-a)} - 1\right)$$

$$- \left(\frac{b-a}{2}\right)^{n-1} \left(\frac{\partial}{\partial a} \mathcal{D}_{n-1}^{\alpha,\beta+2}(a,b)\right) P_{n-1}^{\alpha,\beta+2}\left(2\frac{(t-a)}{(b-a)} - 1\right)$$
$$+ \frac{n-1}{2}\left(\frac{b-a}{2}\right)^{n-2} \mathcal{D}_{n-1}^{\alpha,\beta+2}(a,b) P_{n-1}^{\alpha,\beta+2}\left(2\frac{(t-a)}{(b-a)} - 1\right).$$

To compute the derivative of the determinant, we use the fact that $\mathcal{D}_n^{\alpha,\beta}(a,b)$, can also be expressed as

$$\mathcal{D}_n^{\alpha,\beta}(a,b) = \prod_{j=0}^{n-1} h_j^{\alpha,\beta}(a,b). \tag{4.4}$$

Now,

$$\begin{aligned} h_n^{(\alpha,\beta)}(a,b) &= \int_a^b (b-x)^\alpha (x-a)^\beta \left(p_n^{\alpha,\beta}(x)\right)^2 \, \mathrm{d}x \\ &= \left(\frac{b-a}{2}\right)^{\alpha+\beta+1} \int_{-1}^1 (1-t)^\alpha (t+1)^\beta p_n^{\alpha,\beta}\left[(t+1)\left(\frac{b-a}{2}\right) + a\right] \mathrm{d}t \\ &= \left(\frac{b-a}{2}\right)^{\alpha+\beta+1+2n} h_n^{(\alpha,\beta)}(-1,1) \end{aligned}$$

where $h_n^{\alpha,\beta}(-1,1)$ is given by (1.10). Therefore,

$$\frac{\partial}{\partial a}\mathcal{D}_n^{\alpha,\beta}(a,b) = -\frac{n(\alpha+\beta+n)}{b-a}\mathcal{D}_n^{\alpha,\beta}(a,b).$$

Now putting $a=-1$ and $b=1$ we arrive at the transformed Jacobi polynomials

$$\begin{aligned} p_n^{\alpha,\beta}(t,[\Omega(x)]) = {} & \frac{\mathcal{D}_n^{\alpha,\beta}}{\mathcal{D}_n[\Omega(x)]} P_n^{\alpha,\beta}(t) + A\frac{\mathcal{D}_{n-1}^{\alpha,\beta+2}}{\mathcal{D}_n[\Omega(x)]}\Bigg[P_{n-1}^{\alpha,\beta+2}(t) + \frac{t+1}{\beta+2}(n-1) \\ & \times \left((\alpha+\beta+2+n)P_{n-1}^{\alpha,\beta+2}(t) - (t-1)P_{n-2}^{\alpha+1,\beta+3}(t)\right)\Bigg] \\ & - A^2\frac{\mathcal{D}_{n-2}^{\alpha,\beta+4}}{\mathcal{D}_n[\Omega(x)]}(t+1)^2 P_{n-2}^{\alpha,\beta+4}(t) \end{aligned} \tag{4.5}$$

where $\Omega(x) := (1-x)^\alpha (x+1)^\beta + A\delta'(x+1)$. Also, we have made use of the fact that for the monic Jacobi polynomials

$$\frac{\mathrm{d}}{\mathrm{d}t} P_n^{\alpha,\beta}(t) = n P_{n-1}^{\alpha+1,\beta+1}(t).$$

Note that

$$P_{n-2}^{\alpha,\beta}(t) = A_n^{\alpha,\beta}(t) P_{n-1}^{\alpha,\beta}(t) + B_n^{\alpha,\beta}(t) P_{n-1}^{\alpha+1,\beta}(t) \tag{4.6}$$

where

$$\begin{aligned} A_n^{\alpha,\beta}(t) = {} & \frac{(2n+\alpha+\beta-2)(2n+\alpha+\beta-3)}{2(n-1)(n+\alpha+\beta-1)} \\ & \times \left[\frac{(2n+\alpha+\beta-1)((2n+\alpha+\beta)(2n+\alpha+\beta-2)t+\alpha^2-\beta^2)}{2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)} \right. \\ & \left. - \frac{(n+\alpha)(n+\alpha+\beta)(2n+\alpha+\beta-2)}{(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}\right] \end{aligned}$$

and

$$B_n^{\alpha,\beta}(t) = \frac{(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)(1-t)}{4(n-1)(n+\alpha+\beta-1)(n+\beta-1)(n+\alpha-1)}.$$

We now rearrange the polynomials into a form which will be useful in the next section:

$$
\begin{aligned}
(-1)^n R(n,\alpha,\beta) p_n^{\alpha,\beta}(-t) &= P_n^{\beta,\alpha}(t) \\
&- Ar_1(n,\alpha,\beta)\Bigg[A_{n+1}^{\beta+2,\alpha}(t)P_n^{\beta+2,\alpha}(t) + B_{n+1}^{\beta+2,\alpha}(t)P_n^{\beta+3,\alpha}(t) + \frac{(1-t)}{(\beta+2)}(n-1) \\
&\times \big((\alpha+\beta+2+n)\big[A_{n+1}^{\beta+2,\alpha}(t)P_n^{\beta+2,\alpha}(t) + B_{n+1}^{\beta+2,\alpha}(t)P_n^{\beta+3,\alpha}(t)\big] \\
&- (t+1)\big[A_n^{\beta+3,\alpha+1}(t)\big(A_{n+1}^{\beta+3,\alpha+1}(t)P_n^{\beta+3,\alpha+1}(t) + B_{n+1}^{\beta+3,\alpha+1}(t)P_n^{\beta+4,\alpha+1}(t)\big) \\
&+ B_n^{\beta+3,\alpha+1}(t)\big(A_{n+1}^{\beta+4,\alpha+1}(t)P_n^{\beta+4,\alpha+1}(t) + B_{n+1}^{\beta+4,\alpha+1}(t)P_n^{\beta+5,\alpha+1}(t)\big)\big]\big)\Bigg] \\
&- A^2 r_2(n,\alpha,\beta)(1-t)^2\big[A_n^{\beta+4,\alpha}(t)\big(A_{n+1}^{\beta+4,\alpha}(t)P_n^{\beta+4,\alpha}(t) + B_{n+1}^{\beta+4,\alpha}(t)P_n^{\beta+5,\alpha}(t)\big) \\
&+ B_n^{\beta+4,\alpha}(t)\big(A_{n+1}^{\beta+5,\alpha}(t)P_n^{\beta+5,\alpha}(t) + B_{n+1}^{\beta+5,\alpha}(t)P_n^{\beta+6,\alpha}(t)\big)\big] \qquad (4.7)
\end{aligned}
$$

where we have used

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x)$$

and (4.6).

Here,

$$r_1(n,\alpha,\beta) := \frac{\mathcal{D}_{n-1}^{\alpha,\beta+2}}{\mathcal{D}_n^{\alpha,\beta}} = 2^{-(\alpha+\beta+1)}\frac{\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+2)\Gamma(n)\Gamma(n+\alpha)} \sim \frac{2^{-(\alpha+\beta+1)}n^{2(\beta+1)}}{\Gamma(\beta+2)\Gamma(\beta+1)}$$

$$
\begin{aligned}
r_2(n,\alpha,\beta) := \frac{\mathcal{D}_{n-2}^{\alpha,\beta+4}}{\mathcal{D}_n^{\alpha,\beta}} &= r_1(n,\alpha,\beta)r_1(n-1,\alpha,\beta+2) \\
&\sim \frac{2^{-2(\alpha+\beta+2)}n^{4(\beta+2)}}{\Gamma(\beta+4)\Gamma(\beta+3)\Gamma(\beta+2)\Gamma(\beta+1)}
\end{aligned}
$$

and

$$R(n,\alpha,\beta) := \frac{\mathcal{D}_n[\Omega(x)]}{\mathcal{D}_n^{\alpha,\beta}}.$$

For the transformed Hankel determinant, $\mathcal{D}_n[\Omega(x)]$, observe that

$$
\begin{aligned}
\left[\frac{\partial}{\partial a}\mathcal{D}_{n-1}^{\alpha,\beta+2}(a,b)\right]_{a=-1,\,b=1} &= -(\beta+2)\mathcal{D}_{n-1}^{\alpha,\beta+2}(-1,1) \\
&\times \int_{-1}^{1}\frac{1}{x+1}K_{n-1}(x,x,(1-x)^\alpha(x+1)^{\beta+2})\,\mathrm{d}x
\end{aligned}
$$

which implies

$$\int_{-1}^{1}\frac{1}{x+1}K_{n-1}(x,x,(1-x)^\alpha(x+1)^{\beta+2})\,\mathrm{d}x = \frac{(n-1)(\alpha+\beta+1+n)}{2(\beta+2)}. \qquad (4.8)$$

So from (2.7), we have

$$\mathcal{D}_n[\Omega(x)] = \mathcal{D}_n^{\alpha,\beta} + A\mathcal{D}_{n-1}^{\alpha,\beta+2}\frac{(n-1)(\alpha+\beta+1+n)}{(\beta+2)} - A^2\,\mathcal{D}_{n-2}^{\alpha,\beta+4}. \qquad (4.9)$$

If $j_{1,2}^{\alpha}(n)$ are respectively the coefficients of z^{n-1} and z^{n-2} in $P_n^{\alpha,\beta}(z)$

$$
\begin{aligned}
j_1^{\alpha,\beta}(n) &= -2n\left(\frac{n+\beta}{2n+\alpha+\beta}\right) \\
j_2^{\alpha,\beta}(n) &= 2n(n-1)\left[\frac{1}{4} + \frac{\alpha+n}{2n+\alpha+\beta} + \frac{(\alpha+n-1)(\alpha+n)}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}\right]
\end{aligned}
$$

then, using a similar notation to section 3,

$$R(n,\alpha,\beta)c_1^{\alpha,\beta}(n) = j_1^{\alpha,\beta}(n) + Ar_1(n,\alpha,\beta)\bigg[1 + \frac{(n-1)}{\beta+2}\Big(\big(j_1^{\alpha,\beta+2}(n-1)+1\big)(\alpha+\beta+2+n) - \big(j_1^{\alpha+1,\beta+3}(n-2)\big)\Big)\bigg] - A^2 r_2(n,\alpha,\beta)\big(j_1^{\alpha,\beta+4}(n-2)+2\big)$$

and,

$$R(n,\alpha,\beta)c_2^{\alpha,\beta}(n) = j_2^{\alpha,\beta}(n) + Ar_1(n,\alpha,\beta)\bigg[j_1^{\alpha,\beta+2}(n-1) + \frac{(n-1)}{\beta+2}\Big(\big(j_2^{\alpha,\beta+2}(n-1) + j_1^{\alpha,\beta+2}(n-1)\big)(\alpha+\beta+2+n) - \big(j_2^{\alpha+1,\beta+3}(n-2)-1\big)\Big)\bigg] - A^2 r_2(n,\alpha,\beta)\big(j_2^{\alpha,\beta+4}(n-2) + 2j_1^{\alpha,\beta+4}(n-2)+1\big).$$

5. Large *n* asymptotics

The kernel at the coincident point is the eigenvalue density in random matrix theory. In the limit of large n this quantity tends to the density of zeros of the associated orthogonal polynomials. In this section we evaluate the kernel in the limit of large n for both the Laguerre and Jacobi polynomials. If $p_n(z)$ are the monic orthogonal polynomials, then the kernel is

$$K_n(x,y;w(x)) = \frac{\sqrt{w(x)}\sqrt{w(y)}}{h_{n-1}}\frac{(p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x))}{x-y}. \tag{5.1}$$

Rescaling, $t \to t/4n$, and for large n

$$L_n^{\alpha}(t/4n) \sim (-1)^n\Gamma(n+\alpha+1)(t/4)^{-\alpha/2}J_\alpha(\sqrt{t}).$$

Here we insert the polynomials derived in section 3 into (5.1). Then we scale the variables $x \to x/4n$ and $y \to y/4n$ and take the limit $n \to \infty$. The term which dominates in the product $p_n(x)p_{n-1}(y)$ is the product of two Bessel functions of order $\alpha + 4$. However these cancel with those in the product $p_n(y)p_{n-1}(x)$. The largest non-vanishing term in $p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)$ is

$$2^{2\alpha+7}\frac{\Gamma(n+\alpha+5)\Gamma(n+\alpha+4)}{n^9}(\sqrt{x})^{-\alpha}(\sqrt{y})^{-\alpha} \times (\sqrt{y}J_{\alpha+4}(\sqrt{x})J'_{\alpha+4}(\sqrt{y}) - \sqrt{x}J_{\alpha+4}(\sqrt{y})J'_{\alpha+4}(\sqrt{x})).$$

Therefore,

$$\frac{1}{4n}K_n\left(\frac{x}{4n},\frac{y}{4n}\right) \sim \frac{\sqrt{w\left(\frac{x}{4n}\right)}\sqrt{w\left(\frac{y}{4n}\right)}}{h_{n-1}}2^{2\alpha+7}\frac{\Gamma(n+\alpha+5)\Gamma(n+\alpha+4)}{n^9}x^{-\alpha/2}y^{-\alpha/2} \times\frac{(\sqrt{y}J_{\alpha+4}(\sqrt{x})J'_{\alpha+4}(\sqrt{y}) - \sqrt{x}J_{\alpha+4}(\sqrt{y})J'_{\alpha+4}(\sqrt{x}))}{x-y}. \tag{5.2}$$

Note that $h_{n-1} \sim \Gamma(n-2)\Gamma(n+\alpha+2)$ and

$$\frac{\Gamma(n+\alpha+5)\Gamma(n+\alpha+4)}{\Gamma(n-2)\Gamma(n+\alpha+2)} \sim n^{\alpha+9}.$$

Upon squaring (5.2), and noting that $w(x) = x^\alpha e^{-x} + A\delta'(x)$, we find amongst others, terms like $\delta'(x)f(x,y)$, $\delta'(y)f(x,y)$ and $\delta'(x)\delta'(y)f(x,y)$. With the aid of the distributional identity $\delta'(u)f(u,v) = -\delta(u)\partial_u f(u,v)|_{u=0}$, we find that these terms vanish. Finally, the scaled kernel is

$$\lim_{n\to\infty}\frac{1}{4n}K_n\left(\frac{x}{4n},\frac{y}{4n}\right) = \frac{\big(\sqrt{y}J_{\alpha+4}(\sqrt{x})J'_{\alpha+4}(\sqrt{y}) - \sqrt{x}J_{\alpha+4}(\sqrt{y})J'_{\alpha+4}(\sqrt{x})\big)}{2(x-y)} \tag{5.3}$$

which is the same as the Bessel kernel found in the Laguerre ensemble [11] but with α replaced by $\alpha + 4$. A similar analysis can be performed on the Jacobi kernel. The asymptotic for the Jacobi polynomials, for large n and near 1, is

$$P_n^{\alpha,\beta}\left(\cos\left(\frac{\sqrt{x}}{n}\right)\right) \sim \frac{2^n\Gamma(n+\beta+\alpha+1)\Gamma(n+\alpha+1)}{\Gamma(2n+\beta+\alpha+1)}\left(\frac{2}{\sqrt{x}}\right)^{\alpha} J_\alpha(\sqrt{x}). \tag{5.4}$$

In which case

$$P_n^{\alpha,\beta}\left(-\cos\left(\frac{\sqrt{x}}{n}\right)\right) \simeq (-1)^n\frac{2^n\Gamma(n+\beta+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\beta+\alpha+1)}\left(\frac{2}{\sqrt{x}}\right)^{\beta} J_\beta(\sqrt{x}). \tag{5.5}$$

From the last section, we find, for fixed x and large n

$$\lim_{n\to\infty} A_n^{\alpha,\beta}(x) = 4x - 2$$

and

$$\lim_{n\to\infty} B_n^{\alpha,\beta}(x) = 4(1-x).$$

We now use (4.7) to calculate $K_n(-x,-y)$, then scale $x \to \cos\left(\frac{\sqrt{x}}{n}\right)$ and $y \to \cos\left(\frac{\sqrt{y}}{n}\right)$ and take the limit $n \to \infty$. We first look at the largest terms in n and see if they cancel. If they do, we look at the next largest term and so on. In this case the largest terms are the ones with $r_2(n,\alpha,\beta)$. It can easily be shown that the largest non-vanishing term is the product of $P_n^{\beta+4,\alpha}P_n^{\beta+5,\alpha}$. This leads to

$$\begin{aligned}
&K_n\left(-\cos\left(\frac{\sqrt{x}}{n}\right), -\cos\left(\frac{\sqrt{y}}{n}\right)\right)\\
&\quad\sim \frac{\sqrt{w\left(-\cos\left(\frac{\sqrt{x}}{n}\right)\right)}\sqrt{w\left(-\cos\left(\frac{\sqrt{y}}{n}\right)\right)}}{h_{n-1}^{\alpha,\beta}}\frac{x^2y^2}{4n^8}f(n)2^{2\beta+13}\\
&\quad\times x^{\frac{-(\beta+4)}{2}}y^{\frac{-(\beta+4)}{2}}\left(\frac{\sqrt{y}J'_{\beta+4}(\sqrt{y})J_{\beta+4}(\sqrt{x}) - \sqrt{x}J'_{\beta+4}(\sqrt{x})J_{\beta+4}(\sqrt{y})}{x-y}\right)
\end{aligned}$$

where

$$f(n) = \frac{2^{2n}\Gamma(n+\beta+\alpha+5)\Gamma(n+\beta+\alpha+6)\Gamma(n+\beta+5)\Gamma(n+\beta+6)}{\Gamma(2n+\beta+\alpha+5)\Gamma(2n+\beta+\alpha+6)}$$

Again, dealing with the weight function in a similar way to the Laguerre case, the kernel is

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2n^2}K_n\left(-\cos\left(\frac{\sqrt{x}}{n}\right), -\cos\left(\frac{\sqrt{y}}{n}\right)\right)\\
&\quad= \frac{\sqrt{y}J_{\beta+4}(\sqrt{x})J'_{\beta+4}(\sqrt{y}) - \sqrt{x}J_{\beta+4}(\sqrt{y})J'_{\beta+4}(\sqrt{x})}{2(x-y)}.
\end{aligned} \tag{5.6}$$

This is also similar to the case of the classical Jacobi ensemble [12] with the order of the Bessel function increased by 4.

6. Second-order differential equation for the polynomials

In this section we derive a second-order differential equation with rational coefficients for the transformed polynomials. We follow [6]. We note here that the coefficients of the equations to be derived below also depend on n. The polynomial solutions are sometimes known as semiclassical since the equations are not in the Sturm–Liouville form. For a discussion of the semiclassical polynomials see for example [8].

First we re-express the polynomials obtained in the previous sections in the following form:

$$u_n(t) = a_n(t)y_n(t) + b_n(t)y_n'(t) + c_n(t)y_n''(t) + d_n(t)y_n'''(t) \tag{6.1}$$

where

$$u_n(t) = p_n(t, [w(x) + A\delta'(x-a)])$$
$$y_n(t) = p_n(t, [w(x)])$$

and a_n, b_n c_n and d_n are suitable polynomials in t. In the case of Laguerre and Jacobi polynomials the $y_n(t)$ satisfy a second-order differential equation,

$$y_n''(t) + f_n(t)y_n'(t) + g_n(t)y_n(t) = 0. \tag{6.2}$$

Differentiating (6.1) and (6.2) leads to three more equations:

$$\begin{aligned} u_n'(t) = a_n'(t)y_n(t) &+ [a_n(t) + b_n'(t)]y_n'(t) + [b_n(t) + c_n'(t)]y_n''(t) \\ &+ [c_n(t) + d_n'(t)]y_n'''(t) + d_n(t)y_n''''(t) \end{aligned} \tag{6.3}$$

$$y_n'''(t) + f_n(t)y_n''(t) + [f_n'(t) + g_n(t)]y_n'(t) + g_n'(t)y_n(t) = 0 \tag{6.4}$$

$$y_n''''(t) + f_n(t)y_n'''(t) + [2f_n'(t) + g_n(t)]y_n''(t) + [f_n''(t) + 2g_n'(t)]]y_n'(t) + g_n''(t)y_n(t) = 0. \tag{6.5}$$

Eliminating y'''', y''' and y'' from (6.1) and (6.3) we find

$$u_n'(t) = A(t)y_n(t) + B(t)y_n'(t) \tag{6.6}$$

$$u_n(t) = C(t)y_n(t) + D(t)y_n'(t). \tag{6.7}$$

Now we eliminate $y_n'(t)$ from (6.6) and (6.7) to leave

$$y_n(t) = \frac{D(t)u_n'(t) - B(t)u_n(t)}{A(t)D(t) - B(t)C(t)}. \tag{6.8}$$

Differentiating (6.8) allows us to eliminate $y_n(t)$ leaving an equation of the form

$$\begin{aligned} &D(t)u_n''(t) \\ &+\left(D'(t) - \frac{D(t)(A'(t)D(t) + A(t)D'(t) - B'(t)C(t) - B(t)C'(t))}{A(t)D(t) - B(t)C(t)} + C(t) - B(t)\right)u_n'(t) \\ &-\left(B'(t) - \frac{B(t)(A'(t)D(t) + A(t)D'(t) - B'(t)C(t) - B(t)C'(t))}{A(t)D(t) - B(t)C(t)} + A(t)\right)u_n(t) = 0 \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} A(t) = a_n'(t) &- d_n(t)g_n''(t) - (c_n(t) + d_n'(t) - d_n(t)f_n(t))g_n'(t) - [b_n(t) + c_n'(t) \\ &- 2d_n(t)f_n'(t) - g_n(t)d_n(t) - (c_n + d_n'(t) - d_n(t)f_n(t))f_n(t)]g_n(t) \\ B(t) = a_n(t) &+ b_n'(t) - d_n(t)f_n''(t) - 2d_n(t)g_n'(t) - (c_n(t) + d_n'(t) - d_n(t)f_n(t))(f_n'(t) \\ &+ g_n(t)) - [b_n(t) + c_n'(t) - 2d_n(t)f_n'(t) - g_n(t)d_n(t) \\ &- (c_n(t) + d_n'(t) - d_n(t)f_n(t))f_n(t)]f_n(t) \\ C(t) = a_n(t) &- d_n(t)g_n'(t) - (c_n(t) - d_n(t)f_n(t))g_n(t) \\ D(t) = b_n(t) &- d_n(t)(f_n'(t) + g_n(t)) - (c_n(t) - d_n(t)f_n(t))f_n(t). \end{aligned}$$

Note that (6.9) has rational coefficients

$$\frac{p(t)}{q(t)}u_n''(t) + \frac{r(t)}{s(t)}u_n'(t) + \frac{k(t)}{s(t)}u_n(t) = 0 \tag{6.10}$$

where p, q, r, s and k are polynomials in t. For the transformed Laguerre, up to a t-independent multiplier

$$
\begin{aligned}
u_n(t) = L_n^{\alpha}(t) &+ \left[\frac{Ar_1(n,\alpha)}{n}\left(1+\frac{2t(n-1)}{\alpha+2}\right) - \frac{A^2 r_2(n,\alpha)t}{n}\right] L_n^{\alpha\prime}(t) \\
&- \left[\frac{Ar_1(n,\alpha)}{n}\left(1+\frac{2t(n-2)}{a+2}\right) + \frac{A^2 r_2(n,\alpha)(\alpha+4-n)t}{n(n-1)}\right] L_n^{\alpha\prime\prime}(t) \\
&- \left[\frac{2Ar_1(n,\alpha)t}{(\alpha+2)n} - \frac{A^2(\alpha+3)r_2(n,\alpha)t}{n(n-1)}\right] L_n^{\alpha\prime\prime\prime}(t).
\end{aligned}
$$

Note that the monic $L_n^{\alpha}(t)$ satisfy

$$t L_n^{\alpha\prime\prime}(t) + (\alpha+1-t)L_n^{\alpha\prime}(t) + nL_n^{\alpha}(t) = 0.$$

The coefficients of the differential equation are

$$
\begin{aligned}
p(t) &= p_0 + p_1 t \\
q(t) &= q_1 t \\
r(t) &= r_0 + r_1 t + r_2 t^2 + r_3 t^3 + r_4 t^4 \\
s(t) &= s_2 t^2 + s_3 t^3 + s_4 t^4 \\
k(t) &= k_0 + k_1 t + k_2 t^2 + k_3 t^3
\end{aligned}
\tag{6.11}
$$

which can be found the in the appendix. These were found using the prescription given in (6.2–6.10) and with the aid of *Mathematica*.

We now determine the asymptotic zero density of these polynomials by transforming the second-order differential equation into a Riccati equation. As an example we start with the simpler case of the classical Laguerre polynomials. If $u_n(t) = L_n^{\alpha}(t)$, then

$$t u_n''(t) + (\alpha+1-t)u_n'(t) + nu_n(t) = 0. \tag{6.12}$$

For $t \in \mathbb{C}$

$$\phi(t) := \frac{u_n'(t)}{u_n(t)} = \sum_{k=1}^{n} \frac{1}{t - x_k} \tag{6.13}$$

where the x_k are the zeros of the polynomials. As n tends to ∞

$$\sum_{k=1}^{n} \frac{1}{t-x_k} \simeq \int_E \frac{\sigma(x)}{(t-x)}\,\mathrm{d}x \tag{6.14}$$

where $\sigma(x)$ is the density of zeros, supported on $E \subset \mathbb{R}$, and we shall see later that E, in the cases studied in this paper, will be a single interval. By continuing t to E from above: $t \to s + \mathrm{i}\varepsilon$, $s \in E$, we find

$$\sigma(s) = -\frac{1}{\pi}\Im\phi(s+\mathrm{i}\varepsilon). \tag{6.15}$$

The Riccati equation becomes

$$t\phi'(t) + t\phi^2(t) + (\alpha+1-t)\phi(t) + n = 0. \tag{6.16}$$

Replacing t by $4nt$ in (6.16) and dividing the resulting equation by n, as n tends to ∞ the term with first derivative tends to zero. Therefore, (6.16) reduces to a quadratic equation in $\phi(4nt)$ and the solutions are

$$\phi_{\pm}(4nt) = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{t-1}{t}}. \tag{6.17}$$

In which case the asymptotic density becomes

$$\sigma(4ns) \simeq \frac{1}{2\pi}\sqrt{\frac{1-s}{s}} \qquad 0 < s < 1. \tag{6.18}$$

The same computation can be done for the transformed Laguerre polynomials. Replacing $u_n(t) = p_n(t; x^\alpha e^{-x} + A\delta'(x))$, we have from (6.10)

$$D(t)(\phi'(t) + \phi^2(t)) + M(t)\phi(t) + L(t)n = 0. \tag{6.19}$$

From the appendix, for large n, and fixed t,

$$\begin{aligned} D(t) &\sim -\frac{\kappa}{n} \\ M(t) &\sim \frac{\kappa}{n}\left(1 - \frac{(1+\alpha)}{t}\right) \\ L(t) &\sim -\frac{\kappa}{t} \end{aligned}$$

where $\kappa := 2A^2(2+\alpha)r_2(n,\alpha)$. Replacing t by $4nt$ in (6.19), we see that in the limit $n \to \infty$, again $\phi(4nt)$ satisfies a quadratic equation with solutions

$$\phi_\pm(4nt) = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{t-1}{t}} \tag{6.20}$$

which leads to the classical Laguerre density (6.18). To find the zero density of the transformed Jacobi polynomials, we first express (4.5) in the form of (6.1). Using

$$P_{n-1}^{\alpha,\beta+2}(t) = \Theta_n^{\alpha,\beta}(t)\frac{d}{dt}P_n^{\alpha,\beta}(t) + \Psi_n^{\alpha,\beta}(t)P_n^{\alpha,\beta}(t)$$

where

$$\begin{aligned} \Theta_n^{\alpha,\beta}(t) &= \frac{(1-t^2)(n+\alpha+\beta+1)}{n(2n+\alpha+\beta+1)}\left(\frac{(n+\beta+1)}{(n+a)(1+t)^2} + \frac{1}{(1+t)^2}\right) - \frac{(1-t)}{n(1+t)} \\ \Psi_n^{\alpha,\beta}(t) &= \left(\frac{(n+\beta+1)}{(n+\alpha)(1+t^2)} + \frac{1}{(1+t)^2}\right)\left(\frac{(n+\alpha+\beta+1)}{(2n+\alpha+\beta)}\right) \\ &\quad \times \left(\frac{((2n+\alpha+\beta)t+\beta-\alpha)}{(2n+\alpha+\beta-1)} + \frac{2(n+\alpha)}{(2n+\alpha+\beta+1)}\right) \end{aligned}$$

we find

$$\begin{aligned} a_n(t) &= \Bigg(1 + Ar_1(n,\alpha,\beta)\Bigg(\Psi_n^{\alpha,\beta}(t) + \frac{(t+1)(n-1)}{(\beta+2)}\Bigg((\alpha+\beta+2+n)\Psi_n^{\alpha,\beta}(t) \\ &\quad - \frac{(t-1)}{(n-1)}\Psi_n'^{\alpha,\beta}(t)\Bigg)\Bigg) - A^2r_2(n,\alpha,\beta)(t+1)^2 \\ &\quad \times \left(\Theta_{n-1}^{\alpha,\beta+2}(t)\Psi_n'^{\alpha,\beta}(t) + \Psi_{n-1}^{\alpha,\beta+2}(t)\Psi_n^{\alpha,\beta}(t)\right)\Bigg) \\ b_n(t) &= \Bigg(Ar_1(n,\alpha,\beta)\Bigg(\Theta_n^{\alpha,\beta}(t) + \frac{(t+1)(n-1)}{(\beta+2)}\Bigg((\alpha+\beta+2+n)\Theta_n^{\alpha,\beta}(t) \\ &\quad - \frac{(t-1)}{(n-1)}\left(\Theta_n'^{\alpha,\beta}(t) + \Psi_n^{\alpha,\beta}(t)\right)\Bigg)\Bigg) - A^2r_2(n,\alpha,\beta)(t+1)^2 \\ &\quad \times \left(\Theta_{n-1}^{\alpha,\beta+2}(t)\left(\Theta_n'^{\alpha,\beta}(t) + \Psi_n^{\alpha,\beta}(t)\right) + \Psi_{n-1}^{\alpha\beta+2}(t)\Theta_n^{\alpha,\beta}(t)\right)\Bigg) \\ c_n(t) &= -\left(Ar_1(n,\alpha,\beta)\frac{(t+1)(t-1)}{(b+2)}\Theta_n^{\alpha,\beta}(t) + A^2r_2(n,\alpha,\beta)(t+1)^2\Theta_{n-1}^{\alpha,\beta+2}(t)\Theta_n^{\alpha,\beta}(t)\right). \end{aligned}$$

In this instance $d_n(t) = 0$, and we have kept the same notation for $a_n(t)$, $b_n(t)$ and $c_n(t)$.

The classical Jacobi density can be found in a similar way, however, without any scaling in t

$$\lim_{n\to\infty}\frac{\sigma(s)}{n} = \frac{1}{\pi}\sqrt{1-s^2} \qquad s \in (-1, 1).$$

For large n and fixed t the coefficients of the differential equation simplify to

$$D(t) \sim -\frac{2\chi(1-t^2)}{n^2(1+t)}$$

$$M(t) \sim \frac{\chi(2(\alpha-1)+4(1-\beta+2\alpha)t+12(1+\alpha)t^2+(3+\beta+2\alpha)t^3+(2\alpha+2(2+\beta))t^4)}{n^2(1+4t+6t^2+4t^3+t^4)}$$

$$L(t) \sim -\frac{2\chi}{(1+t)}$$

where $\chi := A^2(2+\beta)r_2(n,\alpha,\beta)$. We have not included the full expressions for D, M and L in the appendix as they are very long; however, they are available on request.

Since $\phi(t)$ is of order n, for large n, the Riccati equation reduces to

$$(t^2-1)\Upsilon^2(t) = 1$$

where $\Upsilon(t) := \lim_{n\to\infty}\left(\frac{\phi(t)}{n}\right)$. So once again we have the classical Jacobi density.

There is an alternative scaling, which changes the density qualitatively, namely, for Laguerre, $\alpha = \mathrm{O}(n)$. So scaling $\alpha \to 4n\alpha$ and $t \to 4nt$, the Riccati equation, in the large n limit, reduces to

$$4t\phi^2(4nt) + 4(\alpha - t)\phi(4nt) + 1 = 0. \tag{6.21}$$

Solving the quadratic equation and upon analytic continuation, the zero density of the classical Laguerre polynomials under this scaling is

$$\lim_{n\to\infty}\sigma(4ns) = \frac{\alpha\delta(s)}{2} + \frac{\sqrt{(s_+-s)(s-s_-)}}{2\pi s} \qquad s_- < s < s_+ \tag{6.22}$$

where

$$s_\pm = \alpha + \tfrac{1}{2} \pm \sqrt{\alpha + 1/4} \qquad \alpha > 0. \tag{6.23}$$

So the density has an absolutely continuous part and a mass point.

Scaling $\alpha \to 4n\alpha$ for the transformed Laguerre polynomials, changes $r_1(n,\alpha n)$ and $r_2(n,\alpha n)$, in such a way that, as $n \to \infty$, both tend to zero, however, with r_2 tending to zero faster than r_1. Therefore from (3.4), the transformed Laguerre polynomials revert to the classical Laguerre polynomials. So we conclude that the zero density is the same as the classical case. Note that this can also be verified directly by computations on the coefficients in the appendix.

For Jacobi we scale $\alpha \to \alpha n$ and $\beta \to \beta n$. In the limit $n \to \infty$, the Riccati equation becomes

$$(1-t^2)\Upsilon^2(t) + (\beta - \alpha - (\alpha+\beta)t)\Upsilon(t) + 1 + \alpha + \beta = 0. \tag{6.24}$$

Therefore, the zero density of classical Jacobi polynomials, under this scaling, is

$$\lim_{n\to\infty}\frac{\sigma(s)}{n} = \frac{\alpha}{2}\delta(1-s) + \frac{\beta}{2}\delta(1+s) + \frac{(2+\alpha+\beta)}{2\pi}\frac{\sqrt{(s_+-s)(s-s_-)}}{1-s^2}$$
$$-1 < s_- < s < s_+ < 1 \qquad \alpha,\beta > 0 \tag{6.25}$$

where

$$s_\pm = \frac{\beta^2 - \alpha^2 \pm 4\sqrt{(1+\alpha)(1+\beta)(1+\alpha+\beta)}}{(2+\alpha+\beta)^2}. \tag{6.26}$$

Again we compare this result with that obtained from our differential equation, (6.10). In this case when we scale α and β to αn and βn, $r_2(n, n\alpha, n\beta)$ is larger than $r_1(n, n\alpha, n\beta)$ for large n. Taking into account the largest terms in D, M and L, from our list of coefficients (available on request), we find, after a long computation, the following quadratic equation satisfied by Υ:

$$\tilde{D}\Upsilon^2 + \tilde{M}\Upsilon + \tilde{L} = 0 \tag{6.27}$$

with $\tilde{D}$, $\tilde{M}$ and $\tilde{L}$ in the following form

$$\tilde{D} := \frac{\beta^2 - 2\beta - 2 - \alpha\beta - 2\alpha - 2\beta^2 t + (\beta^2 + 2\beta + 2 + \alpha\beta + 2\alpha)t^2}{(1+\alpha)^2(1+t)}$$

$$\begin{aligned}\tilde{M} := &\frac{1}{(1+\alpha)^2(1+t)^2}(\alpha^2(2+\beta)(1+t)^2 + 2\alpha(1+\beta)(1+t)(1+\beta(t-1)+t)\\ &+\beta(t-1)(\beta^2(t-1) + 2(1+t) + 2\beta(1+t)))\end{aligned}$$

$$\tilde{L} := -\frac{(1+\alpha+\beta)(\beta^2(t-1) + 2(1+t) + 2\beta(1+t) + \alpha(2+\beta)(1+t))}{(1+\alpha)^2(1+t)^2}.$$

The expressions for $\tilde{D}$, $\tilde{M}$ and $\tilde{L}$ can be simplified, in which case, (6.24) is recovered.

Appendix

The coefficients are:

$$p_0 = A(1+\alpha(1-n)-n)r_1(n,\alpha) + A^2(3+4\alpha+\alpha^2)(2+\alpha)r_2(n,\alpha)$$

$$p_1 = 2An(n-1)r_1(n,\alpha) - A^2(\alpha+1+2n)(\alpha+2)r_2(n,\alpha))$$

$$q_1 = n(n-1)$$

$$\begin{aligned}r_0 = &-A^2(2+\alpha)^2(2+3\alpha+\alpha^2)(3+4\alpha+\alpha^2)(n-1)^2 r_1(n,\alpha)^2\\
&+A^3\big(-(2+\alpha)^2(3+4\alpha+\alpha^2)(n-1)r_1(n,\alpha)\big((1-n)(1+\alpha+n)r_1(n,\alpha)^2\\
&-\ 2(2+\alpha)^2(3+4\alpha+\alpha^2)r_2(n,\alpha)\big)\big)\\
&+A^4\big(-(2+\alpha)^2(3+4\alpha+\alpha^2)(6+5\alpha+\alpha^2)r_2(n,\alpha)\,(2(n-1)\\
&\times\ (1+\alpha+n)r_1(n,\alpha)^2 + (2+\alpha)^2(3+4\alpha+\alpha^2)r_2(n,\alpha))\big)\\
&+A^5(2+\alpha)^2(3+4\alpha+\alpha^2)(6+5\alpha+\alpha^2)^2(1+\alpha+n)r_1(n,\alpha)r_2(n,\alpha)^2\end{aligned}$$

$$\begin{aligned}r_1 = &\,2A^2(1+\alpha)(2+\alpha)^3(n-1)^2(1+\alpha+5n+2\alpha n)r_1(n,\alpha)^2\\
&+A^3\big((2+\alpha)^2(n-1)r_1(n,\alpha)((1-n)(3+9n+6n^2+\alpha^2(3+2n)\\
&+\alpha(6+11n+2n^2))r_1(n,\alpha)^2 - 2(1+\alpha)(2+\alpha)^2(8+22n+\alpha^2(3+2n)\\
&+\ \alpha(11+14n))r_2(n,\alpha))\big) + A^4\big(2(2+\alpha)^3 r_2(n,\alpha)((n-1)\\
&\times\ (11+4\alpha^3+21n+10n^2+\alpha^2(19+8n)+\alpha(26+29n+4n^2))r_1(n,\alpha)^2\\
&+(2+\alpha)^2(3+4\alpha+\alpha^2)(5+2\alpha^2+7n+\alpha(7+3n))r_2(n,\alpha))\big)\\
&+A^5\big((2+\alpha)^4(3+\alpha)(-13-15n-2n^2+\alpha^3(-5+2n)\\
&+\alpha^2(-23+n+2n^2)+\alpha(-31-16n+4n^2))r_1(n,\alpha)r_2(n,\alpha)^2\big)\end{aligned}$$

$$\begin{aligned}r_2 = &-A(1+\alpha)^2(2+\alpha)^2(n-1)^2 n r_1(n,\alpha) + A^2\big((2+\alpha)(n-1)((1-n)(2+24n+36n^3\\
&+\alpha^3(1+8n+4n^2)+\alpha^2(4+34+28n^2)+\alpha(5+50n+56n^2))r_1(n,\alpha)^2\\
&+\ (3+\alpha)(2+3a+\alpha^2)^2 n r_2(n,\alpha))\big) + A^3\big(r_1(n,\alpha)((\alpha^2(2+10n-4n^2)\end{aligned}$$

$$
\begin{aligned}
&+2\alpha(-1-7n+n^2))(n-1)^2(1+\alpha+n)r_1(n,\alpha)^2-(24+78\alpha^2+106n\\
&+182\alpha^2 n-20n^2-166\alpha^2n^2-110n^3-94\alpha^2n^3-6\alpha^4(n-1)(1+2n)\\
&-2\alpha^3(n-1)(18+47n+8n^2)-4\alpha(n-1)(18+77n+44n^2))\\
&\times(2+\alpha)^2r_2(n,\alpha))\big)+A^4\left(-(2+\alpha)r_2(n,\alpha)((1-n)(2\alpha^4(-5-4n+4n^2)\right.\\
&+6(-7-25n-10n^2+8n^3)+\alpha^3(-61-82n+48n^2+8n^3)\\
&+\alpha^2(-134-261n+68n^2+48n^3)+\alpha(-125-337n-14n^2+84n^3))\\
&\times r_1(n,\alpha)^2+(2+\alpha)^2(58+6\alpha^5+142n+70n^2+\alpha^4(51+16n)\\
&+\alpha^3(167+116n+10n^2)+\alpha^2(263+306n+58n^2)\\
&\left.+\ \alpha(199+348n+110n^2))r_2(n,\alpha))\right)+A^5\left(-(2+\alpha)^2(3+\alpha)\right.\\
&\times(-32-45n+2n^2+15n^3+3\alpha^4(-3+2n)+2\alpha^3(-26+6n+7n^2)\\
&+\alpha^2(-109-26n+44n^2+8n^3)+\alpha(-98-77n+38n^2+21n^3))\\
&\left.\times\ r_1(n,\alpha)r_2(n,\alpha)^2\right)+A^6(1+\alpha)^2(2+\alpha)^3\\
&\times(3+\alpha)(2+\alpha^2+3n+n^2+\alpha(3+2n))r_2(n,\alpha)^3
\end{aligned}
$$

$$
\begin{aligned}
r_3=\ &A(1+\alpha)(2+\alpha)^2(n-1)^2n(1+2n)r_1(n,\alpha)+A^2\big(2(2+\alpha)(n-1)n\big(2(n-1)\\
&\times(1+5n+2n^2+\alpha^2(1+2n)+2\alpha(1+3n+n^2))r_1(n,\alpha)^2-(1+\alpha)(2+\alpha)^2\\
&\times(2+a+n)r_2(n,\alpha)\big)\big)+A^3\big(2(n-1)r_1(n,\alpha)(2(n-1)n((1+n)^2(-3+2n)\\
&+\alpha^2(-2-n+2n^2)+\alpha(-5-4n+3n^2+2n^3))r_1(n,\alpha)^2\\
&-(2+\alpha)^2(2+17n+35n^2+6n^3+\alpha^3(1+6n)\\
&+\alpha^2(4+27n+14n^2)+\alpha(5+38n+43n^2+6n^3))r_2(n,\alpha))\big)\\
&+A^4\left(2(2+\alpha)r_2(n,\alpha)(-2(n-1)(-2-17n-8n^2+11n^3+4n^4\right.\\
&+\alpha^3(-1-4n+4n^2)+\alpha^2(-4-21n+10n^2+8n^3)\\
&+\alpha(-5-34n+n^2+18n^3+4n^4))r_1(n,\alpha)^2+(2+\alpha)^2(2\alpha^4+\alpha^3(12+7n)\\
&+2\alpha^2(13+16n+4n^2)+2(4+11n+11n^2+n^3)+\alpha(24+47n+26n^2+2n^3))\\
&\left.\times r_2(n,\alpha))\right)+A^5\left((2+\alpha)^2(-32-63n+4n^2+45n^3+10n^4+\alpha^4(-7+6n)\right.\\
&+2\alpha^3(-22+5n+12n^2)+\alpha^2(-99-44n+78n^2+28n^3)\\
&\left.+\ \alpha(-94-111n+66n^2+69n^3+10n^4))r_1(n,\alpha)r_2(n,\alpha)^2\right)\\
&+A^6\left(-2(1+\alpha)(2+\alpha)^3(2+a+n)(2+\alpha^2+3n+n^2+\alpha(3+2n))r_2(n,\alpha)^3\right)
\end{aligned}
$$

$$
\begin{aligned}
r_4=\ &-2A(2+\alpha)^2(n-1)^2n^2r_1(n,\alpha)+A^2\left((2+\alpha)(n-1)n\left(-4(n-1)n(a+2n)r_1(n,\alpha)^2\right.\right.\\
&\left.\left.+\ (2+\alpha)^2(1+a+2n)r_2(n,\alpha)\right)\right)+A^3\left(4nr_1(n,\alpha)\left(-2(n-1)^3n(1+a+n)\right.\right.\\
&\times r_1(n,\alpha)^2-(2+\alpha)^2r_2(n,\alpha)-\alpha^2(2+\alpha)^2r_2(n,\alpha)-(2+\alpha)^2nr_2(n,\alpha)\\
&+\alpha^2(2+\alpha)^2nr_2(n,\alpha)-(2+\alpha)^2n^2r_2(n,\alpha)+3(2+\alpha)^2n^3r_2(n,\alpha)\\
&\left.\left.+\ \alpha(2+\alpha)^2(n-1)(2+3n)r_2(n,\alpha)\right)\right)+A^5\left(-2(2+\alpha)^2(n-1)\right.\\
&\times\left(2+\alpha^3+10n+13n^2+5n^3+\alpha^2(4+6n)+\alpha(5+16n+10n^2)\right)\\
&\left.\times\ r_1(n,\alpha)r_2(n,\alpha)^2\right)+A^6\left((2+\alpha)^3(1+a+2n)\right.\\
&\left.\times\ (2+\alpha^2+3n+n^2+\alpha(3+2n))r_2(n,\alpha)^3\right)
\end{aligned}
$$

$$
\begin{aligned}
s_2=\ &A(1+\alpha)(2+\alpha)^3(n-1)^2nr_1(n,\alpha)+A^2\left(-(2+\alpha)^2(n-1)n\big((n-1)(1+a+n)\right.\\
&\left.\times\ r_1(n,\alpha)^2+(2+\alpha)^2(3+4a+\alpha^2)r_2(n,\alpha)\big)\right)+A^3\left((2+\alpha)^3(3+\alpha)\right.\\
&\left.\times(n-1)n(1+a+n)r_1(n,\alpha)r_2(n,\alpha)\right)
\end{aligned}
$$

$$
\begin{aligned}
s_3 &= -A(2+\alpha)^2(n-1)^2n(1+a+4n+2an)r_1(n,\alpha) + A^2\big(2(n-1)n\big((2+\alpha)(n-1)\\
&\quad \times (1+a+n)r_1(n,\alpha)^2 + (2+\alpha)^4(1+a+2n)r_2(n,\alpha)\big)\big) + A^3((2+\alpha)^3(n-1)\\
&\quad \times n(1+a+n)(-7+4n+\alpha(-3+2n))r_1(n,\alpha)r_2(n,\alpha))\\
s_4 &= (2+\alpha)^2(n-1)^2n^2 + A\big(2(2+\alpha)(n-1)^2n^2(a+2n)r_1(n,\alpha)\big)\\
&\quad + A^2\left((n-1)n(4(n-1)^2n(1+a+n)r_1(n,\alpha)^2 - \alpha^2(2+\alpha)^2r_2(n,\alpha)\right.\\
&\quad - \left.\alpha(2+\alpha)^2(3+2n)r_2(n,\alpha) - 2(2+\alpha)^2(1+n+n^2)r_2(n,\alpha))\right)\\
&\quad + A^3\left(-2(2+\alpha)(n-1)^2n(1+a+n)(2+a+2n)r_1(n,\alpha)r_2(n,\alpha)\right)\\
&\quad + A^4(2+\alpha)^2(n-1)n(2+\alpha^2+3n+n^2+\alpha(3+2n))r_2(n,\alpha)^2\\
k_0 &= -A^2(2+\alpha)^3(3+4a+\alpha^2)(n-1)^2nr_1(n,\alpha)^2 + A^3\left((1+\alpha)(2+\alpha)^2(n-1)r_1(n,\alpha)\right.\\
&\quad \times \left.\left(-(n-1)^2(1+a+n)r_1(n,\alpha)^2 + 2(6+5a+\alpha^2)^2nr_2(n,\alpha)\right)\right)\\
&\quad + A^4\left(-(2+\alpha)^3(3+4a+\alpha^2)r_2(n,\alpha)(-2(n-1)^2(1+a+n)r_1(n,\alpha)^2\right.\\
&\quad + \left.(6+5a+\alpha^2)^2nr_2(n,\alpha))\right) + A^5\left(-(1+\alpha)(2+\alpha)^2(6+5a+\alpha^2)^2\right.\\
&\quad \times \left.(n-1)(1+a+n)r_1(n,\alpha)r_2(n,\alpha)^2\right)\\
k_1 &= A^2(2+\alpha)^2(n-1)^2n(a+18n+18an+\alpha^2(1+4n))r_1(n,\alpha)^2 + A^3\big((n-1)nr_1(n,\alpha)\\
&\quad \times (16(n-1)^2(1+a+n)r_1(n,\alpha)^2 + 20\alpha(n-1)^2(1+a+n)\\
&\quad \times r_1(n,\alpha)^2 + 6\alpha^2(n-1)^2(1+a+n)r_1(n,\alpha)^2 - 4\alpha^4(2+\alpha)^2\\
&\quad \times (1+n)r_2(n,\alpha) - 8(2+\alpha)^2(2+19n)r_2(n,\alpha) - 2\alpha^3(2+\alpha)^2\\
&\quad \times (13+22n)r_2(n,\alpha) - 4\alpha(2+\alpha)^2(13+67n)r_2(n,\alpha) - 2\alpha^2(2+\alpha)^2\\
&\quad \times (29+84n)r_2(n,\alpha))\big) + A^4\big((2+\alpha)^2r_2(n,\alpha)(-2(n-1)^2\\
&\quad \times (2+21n+19n^2+\alpha^3(1+4n)+\alpha^2(4+25n+4n^2)\\
&\quad + \alpha(5+42n+20n^2))r_1(n,\alpha)^2 + (2+\alpha)^2(3+\alpha)n(8+3\alpha^3+22n\\
&\quad + 2\alpha^2(7+3n)+\alpha(19+24n))r_2(n,\alpha))\big) + A^5(2(2+\alpha)^3(n-1)(1+a+n)\\
&\quad \times (8+22n+\alpha^3(1+n)+\alpha^2(7+10n)+\alpha(14+29n))r_1(n,\alpha)r_2(n,\alpha)^2)\\
&\quad - 2A^6(2+\alpha)^3(6+11a+6\alpha^2+\alpha^3)(2+\alpha^2+3n+n^2+\alpha(3+2n))r_2(n,\alpha)^3\\
k_2 &= -A(1+\alpha)(2+\alpha)^2(n-1)^2n^2r_1(n,\alpha) + A^2\big((2+\alpha)(n-1)n^2(-2(n-1)\\
&\quad \times (14n+2\alpha^2(1+n)+3\alpha(1+4n))r_1(n,\alpha)^2 + (2+\alpha)^2(3+4a+\alpha^2)r_2(n,\alpha))\big)\\
&\quad + A^3\big((n-1)nr_1(n,\alpha)(-20(n-1)^2n(1+a+n)r_1(n,\alpha)^2 - 12\alpha(n-1)^2\\
&\quad \times n(1+a+n)r_1(n,\alpha)^2 + 2\alpha^3(2+\alpha)^2(1+4n)r_2(n,\alpha) + \alpha^2(2+\alpha)^2\\
&\quad \times (7+44n+16n^2)r_2(n,\alpha) + 2(2+\alpha)^2(1+11n+39n^2)r_2(n,\alpha) + \alpha(2+\alpha)^2\\
&\quad \times (7+64n+74n^2)r_2(n,\alpha))\big) + A^4\big((2+\alpha)nr_2(n,\alpha)(2(n-1)^2(10+46n\\
&\quad + 36n^2+4\alpha^3(1+n)+\alpha^2(17+36n+4n^2)+\alpha(23+77n+28n^2))r_1(n,\alpha)^2\\
&\quad - (2+\alpha)^2\big(10+3\alpha^4+34n+46n^2+\alpha^3(19+10n)+\alpha^2(41+52n+10n^2)\\
&\quad + \alpha(35+80n+44n^2)\big)r_2(n,\alpha))\big) + A^5(-(2+\alpha)^2(n-1)(1+a+n)\\
&\quad \times (4+58n+61n^2+\alpha^3(1+4n)+\alpha^2(5+35n+8n^2)+\alpha(8+83n+49n^2))\\
&\quad \times r_1(n,\alpha)r_2(n,\alpha)^2) + A^6(2+\alpha)^3(2+\alpha^2+3n+n^2+\alpha(3+2n))\\
&\quad \times (4+11n+\alpha^2(2+n)+2\alpha(3+4n))r_2(n,\alpha)^3\\
k_3 &= 2A(2+\alpha)^2(n-1)^2n^3r_1(n,\alpha) + A^2((2+\alpha)(n-1)n^2(4(n-1)n(a+2n)r_1(n,\alpha)^2
\end{aligned}
$$

$$- (2+\alpha)^2(1+a+2n)r_2(n,\alpha))) + A^3(4(n-1)n^2 r_1(n,\alpha)(2(n-1)^2$$
$$\times n(1+a+n)r_1(n,\alpha)^2 - \alpha^2(2+\alpha)^2 r_2(n,\alpha) - \alpha(2+\alpha)^2(2+3n)r_2(n,\alpha)$$
$$- (2+\alpha)^2(1+2n+3n^2)r_2(n,\alpha))) + A^4((2+\alpha)nr_2(n,\alpha)(-8\alpha(n-1)^2$$
$$\times n(1+a+n)r_1(n,\alpha)^2 - 4(n-1)^2 n(1+a+n)(3+4n)r_1(n,\alpha)^2$$
$$+ \alpha^3(2+\alpha)^2 r_2(n,\alpha) + \alpha^2(2+\alpha)^2(1+2n)r_2(n,\alpha) + \alpha^2(2+\alpha)^2(3+2n)$$
$$\times r_2(n,\alpha) + \alpha(2+\alpha)^2(1+2n)(3+2n)r_2(n,\alpha) + 2\alpha(2+\alpha)^2(1+n+n^2)$$
$$\times r_2(n,\alpha) + 2(2+\alpha)^2(1+2n)(1+n+n^2)r_2(n,\alpha))) + A^5(2(2+\alpha)^2(n-1)$$
$$\times n(1+a+n)(2+\alpha^2+8n+5n^2+\alpha(3+5n))r_1(n,\alpha)r_2(n,\alpha)^2)$$
$$- A^6(2+\alpha)^3 n(1+a+2n)(2+\alpha^2+3n+n^2+\alpha(3+2n))r_2(n,\alpha)^3$$

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